

Phase Transition in a Traffic Model with Passing

I. Ispolatov¹ and P. L. Krapivsky²

¹*Department of Chemistry, Baker Laboratory, Cornell University, Ithaca, NY 14853*

²*Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215*

We investigate a traffic model in which cars either move freely with quenched intrinsic velocities or belong to clusters formed behind slower cars. In each cluster, the next-to-leading car is allowed to pass and resume free motion. The model undergoes a phase transition from a disordered phase for the high passing rate to a jammed phase for the low rate. In the disordered phase, the cluster size distribution decays exponentially in the large size limit. In the jammed phase, the distribution of finite clusters is independent on the passing rate, but it accounts only for a fraction of all cars; the “excessive” cars form an infinite cluster moving with the smallest velocity. Mean-field equations, describing the model in the framework of Maxwell approximation, correctly predict the existence of phase transition and adequately describe the disordered phase; properties of the jammed phase are studied numerically.

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I. INTRODUCTION

Traffic flows on single-lane roads with no passing exhibit clustering since queues of fast cars accumulate behind slow cars. These clusters form and grow even when car density is small. The initial analysis of cluster formation was carried out in the earlier days of traffic theory [1], and this subject continued growing ever then [2–9]. If passing is introduced, the clusters may stop growing after reaching a certain size. Indeed, previous work [10–12] indicated that after a transient regime a steady state is reached. The models of Refs. [10–12] assume that *any* car in a cluster can pass the leading car and the passing rate is independent on the location of the car within the cluster. This is certainly an oversimplification of the everyday traffic scenarios. The complementary case when only the next-to-leading cars can pass is also an idealization, yet it is closer to reality. Below we show that the latter model is also richer phenomenologically as it undergoes a dynamical phase transition.

We first comment on possible theoretical approaches. A mean-field theory is the primary candidate, and we believe that it may be very good, perhaps even exact, since clustering and passing mix positions and velocities of the cars. The Boltzmann equation approach is an appropriate mean-field scheme, and in our earlier work [10,11] we indeed used it. However, the present model, where only the next-to-leading car is allowed to pass, is significantly more difficult than the model [10,11] where passing was possible for all cars. Indeed, it appears impossible even to write down closed Boltzmann equations for the distribution functions like $P(v, t)$ and $P_m(v, t)$, the density of all clusters moving with velocity v , and the density of clusters of m cars, respectively. Therefore our theoretical analysis is performed in the framework of the Maxwell approach. This scheme simplifies “collision” terms by replacing the actual collision rates, which are proportional to velocity difference of collision partners, by constants. Despite this essentially uncontrolled approximation, the Maxwell approximation is very popular in kinetic theory

[13] and it has already been used in traffic [12].

The important feature of our model is *quenched disorder*, which manifests itself in the random assignment of intrinsic velocities. Road conditions (construction zones, turns, hills, etc.) present another source of quenched randomness in real driving situations [14], which is ignored in our model. Quenched disorder significantly affects characteristics of many-particle systems, especially in low spatial dimensions [15]. This general conclusion applies to the present one-dimensional traffic model as we shall show below.

II. MAXWELL APPROXIMATION

We now formally define the model. Free cars move with *quenched* intrinsic velocities randomly assigned from some distribution $P_0(v)$. When a car or a cluster encounters a slower one, it assumes its velocity and a larger cluster is formed. In every cluster, the next-to-leading car is allowed to pass and resume driving with its intrinsic velocity. The rate of passing is assumed to be a constant. Thus clusters move and aggregate deterministically, while passing is stochastic. The system is initialized by randomly placing single cars and assigning them uncorrelated intrinsic velocities.

Within the Maxwell approach, the joint size-velocity distribution function (the density of clusters of size m moving with velocity v) $P_m(v, t)$ obeys

$$\begin{aligned} \frac{\partial P_m(v, t)}{\partial t} = & \gamma(1 - \delta_{m,1})[P_{m+1}(v, t) - P_m(v, t)] \\ & + \gamma\delta_{m,1}[N(v, t) + P_2(v, t)] - c(t)P_m(v, t) \\ & + \int_v^\infty dv' \sum_{i+j=m} P_i(v', t)P_j(v, t). \end{aligned} \quad (1)$$

Here γ is the passing rate, so terms proportional to γ account for escape, while the rest describes clustering. The escape terms are the same within Boltzmann and

Maxwell approaches, and they are actually *exact*. The collision terms are mean-field by nature, and they are different in the Boltzmann and Maxwell approaches. For instance, in the Boltzmann case, the integral term must involve $v' - v$. Eqs. (1) also contain $c(t)$, the total cluster density

$$c(t) = \sum_{j \geq 1} \int_0^\infty dv P_j(v, t), \quad (2)$$

and $N(v, t)$, the density of clusters in which the next-to-leading car has intrinsic velocity v . This $N(v, t)$ causes the major trouble since it cannot be expressed through $P_j(v, t)$. One might try to close Eqs. (1) by introducing $F_k(v, v', t)$, the density of clusters moving with the velocity v' whose k^{th} car has intrinsic velocity v . Clearly, $N(v, t) = \int_0^v dv' F_2(v, v', t)$, and it appears that equations for $F_k(v, v', t)$ are closed. A more careful look, however, reveals that the governing equation for $F_2(v, v', t)$ includes three-velocity correlators.

Thus, at the first sight, the Boltzmann and Maxwell approaches appear to be equally incapable of providing closed equations for the joint size-velocity distribution function. Still, the Maxwell framework has an advantage that it does provide a closed description on the level of the cluster size distribution. Indeed, integrating Eqs. (1) over velocity and defining $P_m(t) \equiv \int_0^\infty dv P_m(v, t)$, we find that the cluster size distribution $P_m(t)$ obeys

$$\frac{dP_m}{dt} = \gamma[P_{m+1} - P_m] - c P_m + \frac{1}{2} \sum_{i+j=m} P_i P_j \quad (3)$$

for $m \geq 2$, and

$$\frac{dP_1}{dt} = \gamma[P_2 - P_1 + c] - c P_1. \quad (4)$$

Besides this formal derivation of Eqs. (3)–(4) by direct integration of Eqs. (1), it is possible to obtain these equations by enumerating all possible ways in which clusters evolve. For instance, consider Eq. (4). Collisions reduce the density of single cars, and the collision rate is clearly equal to $c(t)$, as it is velocity-independent in the framework of the Maxwell approach. The escape term in Eq. (4) is understood by observing that the rate of return of single cars into the system is equal to

$$\gamma \left[2P_2 + \sum_{j \geq 3} P_j \right] = \gamma [P_2 - P_1 + c].$$

Here $P_2(t)$ is singled out since passing transforms it into two single cars while an escape from larger clusters produces only one freely moving car.

Eqs. (3)–(4) are closed. Mathematically similar equations were investigated previously in the context of the aggregation-fragmentation model [16,17]. Therefore, we merely present essential steps of the analysis. Restricting

ourselves to the steady state and introducing notations $P_m = \gamma F_m$, $c_\infty = \gamma F$, we recast Eqs. (3)–(4) into

$$F F_m = F_{m+1} - F_m + \delta_{m,1} F + \frac{1}{2} \sum_{i+j=m} F_i F_j. \quad (5)$$

These equations should be solved together with the constraints $\sum_{m \geq 1} P_m = c_\infty$ and $\sum_{m \geq 1} m P_m = 1$, i.e.,

$$\sum_{m \geq 1} F_m = F, \quad \sum_{m \geq 1} m F_m = \gamma^{-1}. \quad (6)$$

Note that the sum $\sum_{m \geq 1} m P_m(t)$ is obviously constant due to car conservation. The constant is equal to the initial concentration c_0 as cars were initially unclustered. Here and below we always choose $c_0 = 1$.

As in Ref. [16], we introduce the generating function

$$\mathcal{F}(z) = \sum_{m \geq 1} (z^m - 1) F_m. \quad (7)$$

This generating function obeys

$$\frac{1}{2} \mathcal{F}^2 + \frac{1-z}{z} \mathcal{F} + \frac{(1-z)^2}{z} F = 0, \quad (8)$$

with the solution

$$\mathcal{F}(z) = \frac{z-1}{z} \left\{ 1 - \sqrt{1 - 2zF} \right\}. \quad (9)$$

The steady state solution (9) exists only when the generating function is real for all the $0 \leq z \leq 1$. Hence, we require that $2F \leq 1$. Assuming that this condition is satisfied, we expand the generating function in the powers of z to obtain the steady state concentrations:

$$F_m = \frac{(2F)^m}{2\sqrt{\pi}} \left\{ \frac{\Gamma(m - \frac{1}{2})}{\Gamma(m+1)} - 2F \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m+2)} \right\}. \quad (10)$$

This solution is still incomplete as we have not yet determined F . To find F we use the sum rules (6). The first sum rule is manifestly obeyed, while the second sum rule yields $\sum m F_m = d\mathcal{F}/dz|_{z=1} = 1 - \sqrt{1 - 2F} = \gamma^{-1}$. Thus, $F = \frac{2\gamma-1}{2\gamma^2}$, which translates into $c_\infty = 1 - 1/2\gamma$.

The steady state solution (10) exists for sufficiently high passing rates, $\gamma \geq \gamma_c = 1$. For $\gamma > 1$ and large m , the steady state size distribution simplifies to

$$P_m \simeq C m^{-3/2} \left[1 - (1 - \gamma^{-1})^2 \right]^m, \quad (11)$$

with $C = (4\pi)^{-1/2} \gamma^{-1} (\gamma - 1)^2$. Apart from a power-law prefactor, the size distribution exhibits an exponential decay, $P_m \sim e^{-m/m^*}$, in the large size limit. The characteristic size diverges, $m^* \sim (\gamma - 1)^{-2}$ as the passing rate approaches the critical value $\gamma_c = 1$. In the critical case, the size distribution has a power-law form

$$F_m = \frac{3}{4\sqrt{\pi}} \frac{\Gamma(m - \frac{1}{2})}{\Gamma(m+2)} \sim m^{-5/2}. \quad (12)$$

Let now the passing rate drops below the critical value ($\gamma < \gamma_c$). Since F cannot grow beyond $F_c = 1/2$, it stays constant. Therefore, F_m is given by the same Eq. (12) as in the critical case, and the cluster size distribution reads $P_m = \gamma F_m$. This implies $c_\infty = \gamma/2$, i.e., the sum rule $\sum P_m = c_\infty$ is valid. The second sum rule is formally violated: $\sum m P_m = \gamma \neq 1$, i.e. the cluster size distribution (12) accounts only for the fraction of all the cars present in the system. The only possible explanation is the formation of an infinite cluster that contains all the excessive cars. The second sum rule then shows that $1 - \gamma$ of all the cars in the system are in this infinite cluster.

Thus within the framework of the Maxwell approximation, our traffic model displays a phase transition which separates the disordered and jammed phases. The steady state cluster concentration has different dependence on the passing rate for these two phases:

$$c_\infty = \begin{cases} 1 - 1/2\gamma, & \gamma > 1; \\ \gamma/2, & \gamma < 1. \end{cases} \quad (13)$$

In the disordered phase, the size distribution decays exponentially in the large size limit. In the jammed phase, P_m has a power law tail and in addition there is an infinite cluster which contains the following fraction of cars:

$$I = \begin{cases} 0, & \gamma > 1; \\ 1 - \gamma, & \gamma < 1. \end{cases} \quad (14)$$

This phase transition is similar to phase transitions in driven diffusive systems *without* passing [4–7,9] and to phase transitions in aggregation-fragmentation models [16–18]. Also, the mechanism of the formation of the infinite cluster has a strong formal analogy to Bose-Einstein condensation [6,17].

Turning back to the joint size-velocity distribution (1), we note that the lack of an *exact* expression for $N(v)$ in terms of $P_m(v)$ does not mean the lack of a mean-field relation between these quantities. Indeed, the density $N(v)$ of clusters in which the next-to-leading car has intrinsic velocity v , can be written as

$$N(v) = \int_0^v dv' \sum_{j \geq 2} P_j(v') \frac{C(v)}{\int_{v'}^\infty dv'' C(v'')}. \quad (15)$$

Here $\sum_{j \geq 2} P_j(v')$ is the density of “true” clusters (i.e., freely moving cars are excluded) moving with velocity v' . Then, $C(v) = P_0(v) - P(v)$ is the density of cars with intrinsic velocity v which are currently slowed down, i.e., they are neither single cars, nor cluster leaders. Assuming that the velocities of cars inside clusters are perfectly mixed, $C(v)/\int_{v'}^\infty dv'' C(v'')$ gives the probability density that the next-to-leading car in a true v' -cluster has the velocity v . The product form of Eq. (15) reveals its mean-field nature, which is consistent with the spirit of our theoretical approach. One can verify that Eq. (15) agrees with the sum rule $\int dv N(v) = \sum_{j \geq 2} P_j$, thus providing a useful check of self-consistency.

Although Eqs. (1) with $N(v)$ given by (15) seem very complex even in the steady-state regime, several conclusions can be derived without getting their complete solution. We first simplify Eqs. (1) by introducing auxiliary functions

$$Q_m(v) = \int_v^\infty dv' P_m(v'). \quad (16)$$

By inserting $P_m = -\frac{dQ_m}{dv}$ into the Eqs. (1), integrating resulting equations over v , and using the boundary conditions $Q_m(v = \infty) = 0$, we find

$$\begin{aligned} \gamma [Q_{m+1}(v) - Q_m(v)] - c Q_m(v) + \frac{1}{2} \sum_{i+j=m} Q_i(v) Q_j(v) \\ = \delta_{m1} \gamma q(v), \end{aligned} \quad (17)$$

with

$$q(v) = -Q_1(v) - \int_v^\infty dv' N(v'). \quad (18)$$

Eqs. (17) are almost identical to the Eqs. (3)–(4), the velocity is just a parameter. Consequently, we anticipate qualitatively similar results, $Q_m(v) \sim m^{-3/2} e^{-m/m^*}$, and

$$P_m(v) \sim m^{-1/2} e^{-m/m^*}. \quad (19)$$

with the characteristic size $m^*(v, \gamma)$ dependent on both velocity and passing rate. Our more rigorous generating function analysis, performed along the lines described above, confirms the asymptotic form (19).

III. SIMULATIONS

Now let us examine what conclusions obtained within the Maxwell approach are relevant for the original model. We first re-derive the condition for the phase transition in the complete velocity-dependent form. Let us consider a system of reference with the origin moving with the slowest car. We assume that the system is sufficiently large for the slowest car to have negligible velocity. We compare the total flux of cars clustering behind this slowest car, $\sum m P_m \langle v \rangle_m$, to the rate of escape, γ . Here $\langle v \rangle_m$ is an average velocity of a cluster of size m . When the rate of escape becomes less than the rate of accumulation of the cars, the cluster behind the slowest car (analog of the “infinite cluster” for finite systems) grows to remove the excessive cars from the system. Hence, the phase transition point $\tilde{\gamma}_c$ is defined as

$$\sum_{m \geq 1} m P_m \langle v \rangle_m = \tilde{\gamma}_c. \quad (20)$$

For the Maxwell model, where $\langle v \rangle_m = 1$ for all m , Eq. (20) reduces to $\sum m P_m = \gamma_c = 1$ as obtained above. Since large clusters usually form behind slow cars, $\langle v \rangle_m$

is a decreasing function of the cluster size m . In particular, $\langle v \rangle_m$ is always smaller than the average car velocity $\langle v \rangle$, implying $\tilde{\gamma}_c < 1$.

For a rough estimate of $\langle v \rangle_m$, consider a cluster of m cars and *assume* that intrinsic velocities of the cars in the cluster are independent. The leading car has the minimal velocity, so the size-velocity distribution reads

$$P_m(v) \approx m P_0(v) \left[\int_v^\infty dv' P_0(v') \right]^{m-1} P_m. \quad (21)$$

For concreteness, let us consider intrinsic velocity distributions which behave algebraically near the lower cutoff, $P_0(v) \sim v^\mu$ as $v \rightarrow 0$. Then for large clusters we get

$$P_m(v) \sim P_m \exp(-mv^{\mu+1}). \quad (22)$$

This implies that the average cluster velocity $\langle v \rangle_m$ scales with m according to $\langle v \rangle_m \sim m^{-1/(\mu+1)}$, and hence $\tilde{\gamma}_c \sim \sum m^{\mu/(\mu+1)} P_m$. We conclude that the phase transition does exist in the original model, although its location is shifted towards lower passing rate compared to the Maxwell model prediction. This shift is especially significant for small μ ($\mu > -1$ from the normalization requirement).

To check the relevance of other predictions of the Maxwell approach, we performed molecular dynamics simulations. We place $N = 20000$ single cars onto the ring of length $L = N$, so that the average car density is equal to one. Initial positions and velocities of cars were assigned randomly. We considered linear $P_0(v) = \frac{8}{9}v$ ($0 < v < 3/2$), exponential $P_0(v) = e^{-v}$, and $P_0(v) = (2\pi v)^{-1/2}e^{-v/2}$ velocity distributions, which correspond to $\mu = 1, 0, -1/2$ for the small- v asymptotics. All these three distributions have the average velocity equal to one.

In Fig. 1, we plot $\ln[m^{3/2}P_m]$ vs. m for the above three velocity distributions. We take $\gamma = 1$ which, as we concluded before, lies above the phase transition point $\tilde{\gamma}_c$. We expect the system to be in the disordered phase with P_m being expressed by Eq. (11). For the exponential and $P_0(v) = (2\pi v)^{-1/2}e^{-v/2}$ intrinsic velocity distributions, there is a good agreement with the prediction of the Maxwell model (11); for the linear velocity distribution, there are some deviations for small m , but for large m the agreement is satisfactory. The slopes of the plots decrease with μ . Taking into account that at the point of the phase transition the slope equals to zero, this qualitatively confirms that $\tilde{\gamma}_c$ gets smaller when μ decreases.

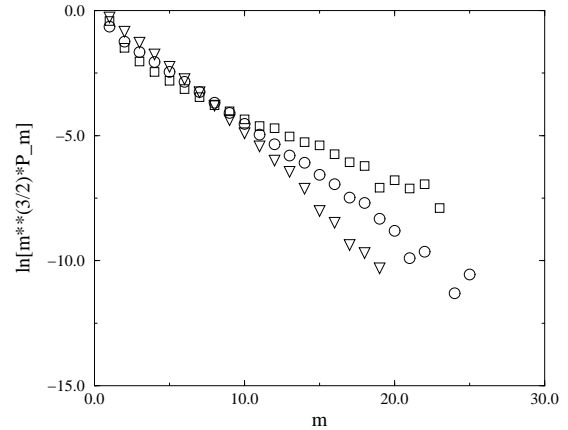


Fig. 1. Plot of $\ln[m^{3/2}P_m]$ vs. cluster size m in the high passing rate regime ($\gamma = 1$) for linear (\square), exponential (\circ), and $P_0(v) = (2\pi v)^{-1/2}e^{-v/2}$ (∇) initial velocity distributions.

Plots of P_m vs. m for intrinsic velocity distributions $P_0(v) = e^{-v}$ and $P_0(v) = (2\pi v)^{-1/2}e^{-v/2}$, with passing rate $\gamma = 0.005$ well below the phase transition point, are shown in Fig. 2a and Fig. 2b, respectively. The cluster size distribution clearly consists of two regions: almost power-law tail for smaller m and several separate peaks for larger m . These peaks correspond to the fluctuating size of the infinite cluster, while the power-law tail describes the regular part of P_m . The apparent exponent τ of the power-law region $P_m \sim m^{-\tau}$ slightly varies for different passing rates and $P_0(v)$, though it remains confined between $3/2$ and 2 . It is definitely different from the value $5/2$, predicted by the Maxwell model (12). The measured values of τ would make the total amount of cars in the system divergent, $\sum m P_m \rightarrow \infty$, so the power-law region ends with an exponential cutoff at large m .

We now comment on the relationship of our model to earlier work. On the mean-field level, our model is similar to the models of Refs. [16,17]. On the level of the process, our model reminds an asymmetric conserved-mass aggregation model (ASCMAM) [17] where clusters undergo asymmetric diffusion, aggregation upon contact, and chipping (single-particle dissociation). Of course, our model is continuum while the ASCMAM is the lattice model. More substantial difference between the two models lies in the nature of randomness – in our model intrinsic velocities are quenched random variables, while in the ASCMAM dynamics is the only source of randomness. Nevertheless, the phenomenology of the two models appears to be quite similar. In particular, the ASCMAM undergoes a phase transition, and in the jammed phase, the cluster size distribution exhibits a power law decay with the exponent close to 2 [17]. We should stress that in the jammed phase, we have not reached a scale-free critical state which must have the exponent $\tau \geq 2$. Maybe quenched randomness does not allow the system to organize itself into a truly normalizable critical state.

Other possible explanation relies on large fluctuations in disordered systems, i.e., our system was not large enough to ensure self-averaging.

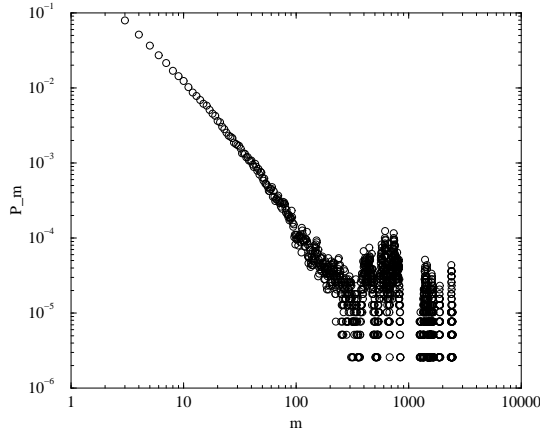


Fig. 2a. Plot of the steady state cluster size distribution P_m in the low passing rate regime ($\gamma = 0.005$) for the exponential initial velocity distribution.

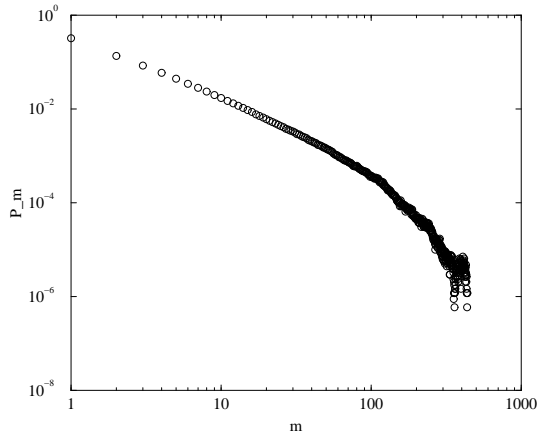


Fig. 2b. Plot of the steady state cluster size distribution P_m in the low passing rate regime ($\gamma = 0.005$) for the $P_0(v) = (2\pi v)^{-1/2} e^{-v/2}$ initial velocity distribution.

IV. CONCLUSION

In this paper, we have investigated the model of traffic that involves clustering and passing of the next-to-leading car. Despite the fact that it is one of the simplest (if not the simplest) possible continuous model of one-lane traffic with passing, the model has rich kinetic behavior. Depending on the passing rate γ the system organizes itself either into disordered phase where density of large clusters is exponentially suppressed, or into

the jammed phase, where the cluster size distribution becomes independent on γ and the infinite cluster is formed. Within the framework of Maxwell approach, which plays the role of the mean-field theory in the present context, we have shown that the model admits an analytical solution. We have argued that the Maxwell approach correctly predicts the existence of the phase transition and adequately describes the properties of the disordered phase which arises when the passing rate is high. For the jammed phase, the Maxwell approach correctly predicts that the system stores excessive cars in the infinite cluster and organizes itself into some kind of a critical state. However, the Maxwell approach cannot quantitatively describe other properties of the jammed phase. It would be interesting to design a more accurate theoretical approach which would allow to probe the characteristics of the low passing rate regime analytically. Some properties of the jammed state appear similar to the properties of the jammed state of a lattice model of Ref. [17] which includes an asymmetric lattice diffusion, aggregation, and fragmentation. It would be interesting to gain a deeper understanding of the relationship between these models, and whether the quenched disorder is the main source of difference.

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- [1] G. F. Newell, *Oper. Res.* **7**, 589 (1959).
 - [2] E. Ben-Naim, P. L. Krapivsky, and S. Redner, *Phys. Rev. E* **50**, 822 (1994).
 - [3] T. Nagatani, *Phys. Rev. E* **51**, 922 (1995).
 - [4] I. Benjamini, P. A. Ferrari, and C. Landim, *Stoch. Proc. Appl.* **61**, 181 (1996).
 - [5] J. Krug and P. A. Ferrari, *J. Phys. A* **29**, L465 (1996).
 - [6] M. R. Evans, *Europhys. Lett.* **36**, 13 (1996); *J. Phys. A* **30**, 5669 (1997).
 - [7] D. V. Kvitarev, D. Chowdhury, and D. E. Wolf, *J. Phys. A* **30**, L221 (1997).
 - [8] W. Knospe, L. Santen, A. Schadschneider, and M. Schreckenberg, *cond-mat/9810184*.
 - [9] For a review, see J. Krug, in *Traffic and Granular Flow II*, eds. D. Wolf and M. Schreckenberg (Springer, Singapore, 1998), p. 285.
 - [10] E. Ben-Naim and P. L. Krapivsky, *Phys. Rev. E* **56**, 6680 (1997).
 - [11] E. Ben-Naim and P. L. Krapivsky, *J. Phys. A* **31**, 8073 (1998).
 - [12] E. Ben-Naim and P. L. Krapivsky, *Phys. Rev. E* **59**, 88 (1999).
 - [13] P. Resibois and M. De Leener, *Classical Kinetic Theory of Fluids* (Wiley, New York, 1977).
 - [14] Influence of such spatial disorder, as opposed to particle-wise disorder included in our model, has been studied by

- several authors, e.g., Z. Csahok and T. Vicsek, J. Phys. A **27**, L591 (1994); O. J. O’Loan, M. R. Evans, and M. E. Cates, Phys. Rev. E **58**, 1404 (1998); G. Tripathy and M. Barma, Phys. Rev. E **58**, 1911 (1998); J. Krug, *cond-mat/9912411*.
- [15] For a review of quenched disorder, see J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
- [16] P. L. Krapivsky and S. Redner, Phys. Rev. E **54**, 3553 (1996).
- [17] S. N. Majumdar, S. Krishnamurthy, and M. Barma, Phys. Rev. Lett. **81**, 3691 (1998); *cond-mat/9908443*.
- [18] R. D. Vigil, R. M. Ziff, and B. Lu, Phys. Rev. B **38**, 942 (1988).